On supersymmetric quantum mechanics in curved spaces

1997 J. Phys. A: Math. Gen. 30 7427

(http://iopscience.iop.org/0305-4470/30/21/016)

J. Phys. A: Math. Gen. **30** (1997) 7427–7436. Printed in the UK PII: S0305-4470(97)83728-6

On supersymmetric quantum mechanics in curved spaces

N Debergh†

Theoretical and Mathematical Physics, Institute of Physics, B.5, University of Liège, B-4000-Liège 1, Belgium

Received 28 April 1997, in final form 4 July 1997

Abstract. Supersymmetric quantum mechanics is considered in one-, two- and three-dimensional spaces characterized by a non-vanishing constant curvature. In particular, a basic superalgebra is pointed out. Interactions such as those of oscillator-like or Calogero type are analysed in this context with respect to their spectra and associated accidental degeneracies.

1. Introduction

Originally coming from particle physics developments, supersymmetric quantum mechanics (SSQM) has received much attention in the recent literature [1] since Witten's contribution [2]. As is well known, it is characterized by the existence of N self-adjoint operators Q_a ($a = 1, 2, \ldots, N$)—the supercharges—each of them being the square root of the Hamiltonian H. More precisely, the Lie superalgebra subtending SSQM is that associated with the following anticommutation and commutation relations:

$$\{Q_a, Q_b\} = 2\delta_{ab} H \tag{1}$$

$$[H, Q_a] = 0 \qquad a = 1, 2, \ldots, N. \tag{2}$$

Let us already notice that we will limit ourselves to the case $N = 2$ corresponding, in particular, to the interactions we will be interested in.

Some known systems are self-supersymmetric while the others need a supersymmetrization procedure. Among the first ones, we find the free context. Indeed, the (D-dimensional) free Schrödinger Hamiltonian is

$$H = \tfrac{1}{2}\boldsymbol{p}^2 \tag{3}$$

and is generated by the two supercharges

$$Q_1 = \frac{1}{\sqrt{2}}(\boldsymbol{p} \cdot \boldsymbol{\varphi}^{(1)}) \qquad Q_2 = \frac{1}{\sqrt{2}}(\boldsymbol{p} \cdot \boldsymbol{\varphi}^{(2)}) \tag{4}$$

where the matrices $\varphi_j^{(1)}$ and $\varphi_j^{(2)}$ ($j = 1, 2, \ldots, D$) satisfy

$$\{\varphi_j^{(1)}, \varphi_k^{(1)}\} = \{\varphi_j^{(2)}, \varphi_k^{(2)}\} = 2\delta_{jk} \tag{5}$$

$$\{\varphi_j^{(1)}, \varphi_k^{(2)}\} = 2\Xi_{jk} \tag{6}$$

Ξ being an antisymmetric tensor. In fact, it has been shown previously [3] that the quantities $\varphi_j^{(1)}$ and $\varphi_j^{(2)}$ are the generators of the superalgebra $su(2^{(D-1)/2}/2^{(D-1)/2})$ if D is odd and

† Chercheur, Institut Interuniversitaire des Sciences Nucléaires, Bruxelles.

$su(2^{(D/2)-1}/2^{(D/2)-1})$ if D is even. Among the second category, i.e. the systems needing a supersymmatrization, we can recognize f.i. the harmonic oscillator-like interaction [4]. More generally, that is to say for an arbitrary interaction, the supercharges can be written as

$$Q_1 = \frac{1}{\sqrt{2}}(\boldsymbol{p}\cdot\boldsymbol{\varphi}^{(1)} + \boldsymbol{W}\cdot\boldsymbol{\varphi}^{(2)}) \qquad Q_2 = \frac{1}{\sqrt{2}}(\boldsymbol{p}\cdot\boldsymbol{\varphi}^{(2)} - \boldsymbol{W}\cdot\boldsymbol{\varphi}^{(1)}) \tag{7}$$

where $\boldsymbol{W}(\boldsymbol{x})$ is called the superpotential. Through (1), they lead to the Hamiltonian

$$H = \tfrac{1}{2}\boldsymbol{p}^2 + \tfrac{1}{2}\boldsymbol{W}^2 + W_k p_j \Xi_{jk} - \tfrac{1}{2}\mathrm{i}\partial_j W_k(\Xi_{jk} + \tfrac{1}{2}[\varphi_j^{(1)}, \varphi_k^{(2)}]). \tag{8}$$

All these considerations are relevant of a *flat* space. Now, it is well known that a way to put in evidence *nonlinear* algebras [5] is to leave such a flat space and go to a *curved* one (characterized by $\lambda > 0$ if it is a sphere, by $\lambda < 0$ if it is a hyperboloïd). If it is true at the level of algebras, we can reasonably hope that it is also true at the level of superalgebras such as that recalled in (1) and (2). It is the purpose of this paper to analyse this situation. More precisely, we will show that the free case as well as the corresponding interacting contexts are supersymmetric in a curved space and imply a generalized SSQM-superalgebra.

The contents are then as follows. In section 2, we introduce the generalized SSQM-superalgebra on the free case example and extend it to the context of an arbitrary interaction. The spectra and eigenfunctions of the new associated Hamiltonians are found in section 3 for the oscillator-like interaction in one, two and three space dimensions and for the Calogero interaction in three space dimensions. The degeneracies present in these spectra are studied and explained in section 4. We also include an appendix recalling the main steps of the supersymmetric factorization method.

2. The generalized SSQM-superalgebra

The free Hamiltonian on a D-dimensional curved space is [6]

$$H = \tfrac{1}{2}(\boldsymbol{\pi}^2 + \lambda\boldsymbol{L}^2) \tag{9}$$

where $\boldsymbol{\pi}$ refers to a new Hermitian momentum defined by

$$\boldsymbol{\pi} = \boldsymbol{p} + \tfrac{1}{2}\lambda\boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{p}) + \tfrac{1}{2}\lambda(\boldsymbol{p}\cdot\boldsymbol{x})\boldsymbol{x} \tag{10}$$

and $\boldsymbol{L}$ is the usual angular momentum

$$L_{jk} = x_j p_k - x_k p_j \qquad j, k = 1, 2, \ldots, D.$$

Moreover, in order to fix our ideas, we consider the strictly positive curvature λ (i.e. we are working on the D-sphere).

The Hamiltonian (9) can be considered as a generalization of the Hamiltonian (3) (which is recovered when $\lambda \to 0$). By analogy with the operators (4), we propose to define the generalized supercharges by

$$Q_1 = \frac{1}{\sqrt{2}}(\boldsymbol{\pi}\cdot\boldsymbol{\varphi}^{(1)}) \qquad Q_2 = \frac{1}{\sqrt{2}}(\boldsymbol{\pi}\cdot\boldsymbol{\varphi}^{(2)}). \tag{11}$$

Noticing that

$$[\pi_j, \pi_k] = \mathrm{i}\lambda L_{jk} \tag{12}$$

it is easy to see that

$$Q_1^2 = \tfrac{1}{2}\boldsymbol{\pi}^2 + \tfrac{1}{8}\mathrm{i}\lambda L_{jk}[\varphi_j^{(1)}, \varphi_k^{(1)}] \tag{13}$$

$$Q_2^2 = \tfrac{1}{2}\boldsymbol{\pi}^2 + \tfrac{1}{8}\mathrm{i}\lambda L_{jk}[\varphi_j^{(2)}, \varphi_k^{(2)}] \tag{14}$$

$$\{Q_1, Q_2\} = \tfrac{1}{4}\mathrm{i}\lambda L_{jk}[\varphi_j^{(1)}, \varphi_k^{(2)}]. \tag{15}$$

Concentrating on the dimensions $D = 1, 2$ and 3, we can, more precisely, realize the quantities $\varphi_j^{(1)}$ and $\varphi_j^{(2)}$ in terms of the Pauli matrices as follows [4]:

$$\varphi_1^{(1)} = \sigma_1 \qquad \varphi_1^{(2)} = \sigma_2 \qquad \text{if } D = 1 \tag{16}$$

$$\varphi_1^{(1)} = \sigma_1 \qquad \varphi_2^{(1)} = \sigma_2 \qquad \varphi_1^{(2)} = \sigma_2 \qquad \varphi_2^{(2)} = -\sigma_1 \qquad \text{if } D = 2 \tag{17}$$

$$\left.\begin{array}{lll} \varphi_1^{(1)} = \sigma_1 \otimes \sigma_1 & \varphi_2^{(1)} = \sigma_2 \otimes \sigma_1 & \varphi_3^{(1)} = \sigma_3 \otimes \sigma_1 \\ \varphi_1^{(2)} = \sigma_1 \otimes \sigma_2 & \varphi_2^{(2)} = \sigma_2 \otimes \sigma_2 & \varphi_3^{(2)} = \sigma_3 \otimes \sigma_2 \end{array}\right\} \qquad \text{if } D = 3 \tag{18}$$

in agreement with (5) and (6). We can then rewrite relations (13)–(15) as

$$\{Q_1, Q_1\} = \{Q_2, Q_2\} = 2H - \lambda \boldsymbol{J}^2 + \tfrac{1}{8}\lambda D(D-1) \tag{19}$$

$$\{Q_1, Q_2\} = 0 \tag{20}$$

where (for $D = 2, 3$)

$$\boldsymbol{J} = \boldsymbol{L} + \tfrac{1}{2}\boldsymbol{\sigma}. \tag{21}$$

Such results imply that

$$[H, Q_1] = [H, Q_2] = 0 \tag{22}$$

$$[H, \boldsymbol{J}] = 0 \tag{23}$$

$$[\boldsymbol{J}, Q_1] = [\boldsymbol{J}, Q_2] = 0 \tag{24}$$

and evidently

$$[J_j, J_k] = \mathrm{i}\varepsilon_{jkl} J_l. \tag{25}$$

The set (19), (20), (22)–(25) imply that we have two *odd* operators (Q_1 and Q_2) and $\frac{1}{2}(D^2 - D + 2)$ *even* ones (H and $\boldsymbol{J}$). These are characteristics of a generalized SSQM-superalgebra. They reduce to equations (1) and (2) if $\lambda \to 0$ ($\boldsymbol{J}$ disappears as it should in the limit of a vanishing λ). Let us also notice that, in the case $D = 1$, the usual SSQM-superalgebra is recovered whether λ is present or not.

If we now turn to the interacting context associated with

$$Q_1 = \frac{1}{\sqrt{2}}(\boldsymbol{\pi} \cdot \boldsymbol{\varphi}^{(1)} + \boldsymbol{W} \cdot \boldsymbol{\varphi}^{(2)}) \qquad Q_2 = \frac{1}{\sqrt{2}}(\boldsymbol{\pi} \cdot \boldsymbol{\varphi}^{(2)} - \boldsymbol{W} \cdot \boldsymbol{\varphi}^{(1)}) \tag{26}$$

by analogy with (7), we are led through the relation (19) to

$$\begin{aligned} H = {} & \tfrac{1}{2}\boldsymbol{\pi}^2 + \tfrac{1}{2}\boldsymbol{W}^2 + \tfrac{\mathrm{i}}{8}\lambda L_{jk}[\varphi_j^{(1)}, \varphi_k^{(1)}] + \tfrac{1}{2}\{\pi_j, W_k\}\Xi_{jk} - \tfrac{1}{4}[\pi_j, W_k][\varphi_k^{(2)}, \varphi_j^{(1)}] \\ & + \tfrac{1}{2}\lambda \boldsymbol{J}^2 - \tfrac{1}{16}\lambda D(D-1) \end{aligned} \tag{27}$$

and the remaining relations of the generalized SSQM-superalgebra are still available.

3. Spectrum and eigenfunctions of the generalized Hamiltonian

Let us concentrate on systems submitted to central forces, i.e. described by superpotentials of the form

$$\boldsymbol{W}(\boldsymbol{x}) = V(r)\boldsymbol{x} \qquad r = |\boldsymbol{x}|. \tag{28}$$

The generalized Hamiltonian (27) can then be written as

$$\begin{aligned} H^{(D)} = {} & \frac{1}{2}\boldsymbol{p}^2 - \mathrm{i}\lambda D \boldsymbol{x} \cdot \boldsymbol{p} - \frac{\mathrm{i}}{2}\lambda^2 (D+2) r^2 \boldsymbol{x} \cdot \boldsymbol{p} + \lambda(\boldsymbol{x} \cdot \boldsymbol{p})^2 + \frac{1}{2}\lambda^2 r^2 (\boldsymbol{x} \cdot \boldsymbol{p})^2 \\ & - \frac{\lambda^2}{8}(D+1)(D+3) r^2 + \frac{1}{2} V^2 r^2 + \frac{\mathrm{i}\lambda}{8} L_{jk}[\varphi_j^{(1)}, \varphi_k^{(1)}] + V x_k p_j \Xi_{jk} \end{aligned}$$

$$+\frac{\mathrm{i}}{4}\left(\delta_{jk}V + \lambda x_j x_k V + \frac{1}{r}x_j x_k V' + \lambda r x_j x_k V'\right)[\varphi_k^{(2)}, \varphi_j^{(1)}]$$
$$+\frac{\lambda}{2}\boldsymbol{J}^2 - \frac{\lambda D}{16}(3+5D) \tag{29}$$

where V' stands for the derivative of $V(r)$ with respect to r. Using the spherical coordinates for which

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta \tag{30}$$

and

$$\boldsymbol{x}\cdot\boldsymbol{p} = -\mathrm{i}r\frac{\partial}{\partial r} \tag{31}$$

we respectively obtain for $D=1$, $D=2$ and $D=3$

$$H^{(1)} = -\frac{1}{2}(1+\lambda r^2)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\lambda r(1+\lambda r^2)\frac{\mathrm{d}}{\mathrm{d}r} - \lambda^2 r^2 + \frac{1}{2}V^2 r^2 - \frac{\lambda}{2}$$
$$+\frac{1}{2}\sigma_3(1+\lambda r^2)(V + rV') \tag{32a}$$

$$H^{(2)} = -\frac{1}{2}(1+\lambda r^2)^2\frac{\partial^2}{\partial r^2} - \frac{1}{2r}(1+\lambda r^2)(1+5\lambda r^2)\frac{\partial}{\partial r} - \frac{1}{2r^2}(1+\lambda r^2)\Delta_\theta$$
$$-\frac{15\lambda}{8}\left(\frac{4}{5}+\lambda r^2\right) + \frac{1}{2}V^2r^2 + VL_3 + V\left(1+\frac{1}{2}\lambda r^2\right)\sigma_3 + \frac{1}{2}r(1+\lambda r^2)V'\sigma_3 \tag{32b}$$

and

$$H^{(3)} = -\frac{1}{2}(1+\lambda r^2)^2\frac{\partial^2}{\partial r^2} - \frac{1}{r}(1+\lambda r^2)(1+3\lambda r^2)\frac{\partial}{\partial r} - \frac{1}{2r^2}(1+\lambda r^2)\Delta_\theta - 3\lambda(1+\lambda r^2)$$
$$+\frac{1}{2}V^2r^2 + V\boldsymbol{L}\cdot\boldsymbol{\sigma}\otimes\sigma_3 + \frac{1}{2}V(3+\lambda r^2)I\otimes\sigma_3 + \frac{1}{2}r(1+\lambda r^2)V'I\otimes\sigma_3. \tag{32c}$$

In order to put in evidence the corresponding eigenvalues E and eigenfunctions ψ satisfying

$$H^{(D)}\psi = E\psi \tag{33}$$

we successively realize the factorizations

$$\psi = R(r)Y(\theta,\ldots) \tag{34}$$

and

$$R(r) = \Lambda(r)\frac{r^{-(1/2)(D-1)}}{1+\lambda r^2} \tag{35}$$

with [7]

$$\Lambda(\theta)Y(\theta,\ldots) = -l(l+D-2)Y(\theta,\ldots). \tag{36}$$

Finally, we consider the following change of variables

$$r = \frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda}y) \tag{37}$$

and the associated factorization

$$\Lambda = \frac{1}{\cos(\sqrt{\lambda}y)}\chi(y). \tag{38}$$

We then obtain equation (33) on the more usual forms

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{1}{2\lambda}V^2\tan^2(\sqrt{\lambda}y)+\frac{\sigma_3}{2\cos^2(\sqrt{\lambda}y)}\left(V+\frac{1}{2\sqrt{\lambda}}\sin(2\sqrt{\lambda}y)\frac{\mathrm{d}V}{\mathrm{d}y}\right)-E\right]\chi=0 \tag{39a}$$

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\lambda(l^2-\frac{1}{4})}{2\tan^2(\sqrt{\lambda}y)}+\frac{1}{2}V\left(\frac{V}{\lambda}+\sigma_3\right)\tan^2(\sqrt{\lambda}y)-\frac{\lambda}{4}+\frac{\lambda l^2}{2}+V\sigma_3+Vm\right.$$
$$\left.+\frac{1}{2}V'\sigma_3(1+\tan^2(\sqrt{\lambda}y))\frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda}y)-E\right]\chi=0 \tag{39b}$$

and

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\lambda l(l+1)}{2\tan^2(\sqrt{\lambda}y)}+\frac{1}{2}V\left(\frac{V}{\lambda}+l\otimes\sigma_3\right)\tan^2(\sqrt{\lambda}y)-\frac{\lambda}{2}+\frac{\lambda}{2}l(l+1)+\frac{3}{2}Vl\otimes\sigma_3\right.$$
$$\left.+Vkl\otimes\sigma_3+\frac{1}{2}V'l\otimes\sigma_3(1+\tan^2(\sqrt{\lambda}y))\frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda}y)-E\right]\chi=0 \tag{39c}$$

for $D=1$, $D=2$ and $D=3$, respectively. Note that m is the eigenvalue of L_3 and k is the eigenvalue of $\boldsymbol{L}\cdot\boldsymbol{\sigma}(k=l$ if $j=l+\frac{1}{2}$, $k=-l-1$ if $j=l-\frac{1}{2})$.

Let us now turn to some specific interactions and let us apply in each case the factorization based on SSQM characteristics as recalled in the appendix in order to put in evidence the eigenvalues and eigenfunctions of $H^{(D)}$.

3.1. The $D=1$ oscillator-like interaction

In this case, the function $V(r)$ introduced in (28) is simply the angular frequency ω. Equation (39a) is then

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\omega^2}{2\lambda}\tan^2(\sqrt{\lambda}y)+\frac{\sigma_3\omega}{2\cos^2(\sqrt{\lambda}y)}-E\right]\chi=0. \tag{40}$$

It is thus a usual supersymmetric equation, i.e. of Witten type [2] (cf equation (8)) characterized by the superpotential

$$W(y)=\frac{\omega}{\sqrt{\lambda}}\tan(\sqrt{\lambda}y). \tag{41}$$

Evidently, in the limit $\lambda\to 0$, we recover $W(y)=\omega y$ as expected. Applying the method developed in the appendix, we successfully obtain for the (-1)-eigenvalue of σ_3

$$W_n(y)=\left(n\sqrt{\lambda}+\frac{\omega}{\sqrt{\lambda}}\right)\tan(\sqrt{\lambda}y) \tag{42}$$

$$E_n^-=n\left(1+\frac{\lambda}{2}n\right)\omega \tag{43}$$

and

$$\chi_n^-\sim(\cos(\sqrt{\lambda}y))^{\omega/\lambda}{}_2F_1\left(\frac{\omega}{\lambda}+\frac{n}{2},-\frac{n}{2};\frac{1}{2};\sin^2(\sqrt{\lambda}y)\right) \tag{44}$$

in terms of the hypergeometric function ${}_2F_1(a,b;c;z)$ satisfying [8]

$$\left(z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c-(a+b+1)z)\frac{\mathrm{d}}{\mathrm{d}z}-ab\right){}_2F_1(a,b;c;z)=0. \tag{45}$$

The supersymmetric properties and, in particular, the double degeneracy [2] of the non-vanishing energies imply that, for the $(+1)$-eigenvalue of σ_3, we have

$$E_n^+ = (n+1)\left(1+\frac{\lambda}{2}(n+1)\right)\omega \tag{46}$$

and

$$\chi_n^+ \sim (\cos(\sqrt{\lambda}y))^{(\omega/\lambda)+1}\sin(\sqrt{\lambda}y)\,{}_2F_1\left(\frac{\omega}{\lambda}+\frac{n+3}{2}, \frac{-n+1}{2}; \frac{3}{2}; \sin^2(\sqrt{\lambda}y)\right). \tag{47}$$

Let us notice that in the hyperboloïd context ($\lambda < 0$) where, essentially, trigonometric functions have to be replaced by hyperbolic ones, the superpotential corresponding to (41) is the Pöschl–Teller one [9].

3.2. *The $D = 2$ oscillator-like interaction*

Replacing V by ω in (39*b*), we obtain

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\lambda\left(l^2-\frac{1}{4}\right)}{2\mathrm{tg}^2(\sqrt{\lambda}y)}+\frac{\omega}{2}\left(\frac{\omega}{\lambda}+\sigma_3\right)\tan^2(\sqrt{\lambda}y)-\frac{\lambda}{4}+\frac{\lambda l^2}{2}+\omega\sigma_3+\omega m-E\right]\chi = 0. \tag{48}$$

Applying once again the suypersymmetric factorization as mentioned in the appendix, we find

$$W_n(y) = \left(\sqrt{\lambda}\left(n+\frac{1}{2}(\varepsilon+1)\right)+\frac{\omega}{\sqrt{\lambda}}\right)\tan(\sqrt{\lambda}y)-\sqrt{\lambda}\left(n+l+\frac{1}{2}\right)\cot(\sqrt{\lambda}y) \tag{49}$$

$$E_n = \frac{\lambda}{2}\left(2n+l+\frac{1}{2}(\varepsilon+1)\right)\left(2n+l+\frac{3}{2}+\frac{\varepsilon}{2}\right)+\omega(2n+l+m+\varepsilon+1) \tag{50}$$

$$\chi_n \sim (\cos(\sqrt{\lambda}y))^{(\omega/\lambda)+((\varepsilon+1)/2)}(\sin(\sqrt{\lambda}y))^{l+(1/2)} \times {}_2F_1\left(\frac{\omega}{\lambda}+n+l+1+\frac{\varepsilon}{2}, -n; l+1; \sin^2(\sqrt{\lambda}y)\right) \tag{51}$$

in terms of $\varepsilon = \pm 1$, the eigenvalues of σ_3.

3.3. *The $D = 3$ oscillator-like interaction*

What we have to do, also in this case, is to replace V by ω in (39*c*) so we have

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\lambda l(l+1)}{2\tan^2(\sqrt{\lambda}y)}+\frac{\omega}{2}\left(\frac{\omega}{\lambda}+l\otimes\sigma_3\right)\tan^2(\sqrt{\lambda}y)-\frac{\lambda}{2}+\frac{\lambda}{2}l(l+1) + \frac{3}{2}\omega l\otimes\sigma_3+\omega k l\otimes\sigma_3-E\right]\chi = 0. \tag{52}$$

The associated superpotentials, energies and eigenfunctions are respectively

$$W_n(y) = \left(\frac{\sqrt{\lambda}}{2}(\varepsilon+1+2n)+\frac{\omega}{\sqrt{\lambda}}\right)\tan(\sqrt{\lambda}y)-\sqrt{\lambda}(n+l+1)\cot(\sqrt{\lambda}y) \tag{53}$$

$$E_n = \frac{\lambda}{2}\left(2n+l+\frac{1}{2}(\varepsilon+1)\right)\left(2n+l+\frac{5}{2}+\frac{\varepsilon}{2}\right)+\omega\left(2n+l+\varepsilon k+\frac{3}{2}(\varepsilon+1)\right) \tag{54}$$

and

$$\chi_n \sim (\cos(\sqrt{\lambda}y))^{(\omega/\lambda)+((\varepsilon+1)/2)}(\sin(\sqrt{\lambda}y))^{l+1} \times {}_2F_1\left(\frac{\omega}{\lambda}+n+l+\frac{3}{2}+\frac{\varepsilon}{2}, -n; l+\frac{3}{2}; \sin^2(\sqrt{\lambda}y)\right). \tag{55}$$

3.4. *The $D = 3$ Calogero interaction*

This interaction corresponds to the superpotential (28) where [10]

$$V(r) = \omega + \frac{\nu}{r^2} \qquad l+\frac{1}{2} \geqslant \nu \geqslant -l-\frac{1}{2}. \tag{56}$$

With the change of variables (37), it is rewritten as

$$V(y) = \omega + \frac{\lambda\nu}{\tan^2(\sqrt{\lambda}y)} \tag{57}$$

implying that equation (39*c*) reduces to

$$\Bigg[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{\tan^2(\sqrt{\lambda}y)}\left(\frac{\lambda}{2}l(l+1)+\frac{1}{2}\lambda\nu^2+\frac{1}{2}\lambda\nu l\otimes\sigma_3+\lambda\nu k l\otimes\sigma_3\right) + \tan^2(\sqrt{\lambda}y)\left(\frac{\omega^2}{2\lambda}+\frac{\omega}{2}l\otimes\sigma_3\right)+\omega\nu-\frac{\lambda}{2}+\frac{\lambda}{2}l(l+1) + \left(\omega k+\frac{3}{2}\omega-\frac{1}{2}\nu\lambda\right)l\otimes\sigma_3 - E\Bigg]\chi = 0. \tag{58}$$

The supersymmetric factorization leads to

$$W_n(y) = \left(\frac{\sqrt{\lambda}}{2}(\varepsilon+1+2n)+\frac{\omega}{\sqrt{\lambda}}\right)\tan(\sqrt{\lambda}y) - \sqrt{\lambda}(n+l+1+\nu\varepsilon)\cot(\sqrt{\lambda}y) \tag{59}$$

$$E_n = \frac{\lambda}{2}\left(2n+l+\frac{1}{2}(\varepsilon+1)\right)\left(2n+l+\frac{5}{2}+\frac{\varepsilon}{2}\right) +\omega\left(2n+l+\varepsilon k+\frac{3}{2}(\varepsilon+1)\right)+\omega\nu(\varepsilon+1)+\lambda\nu\left(2\varepsilon n+\frac{1}{2}(\varepsilon+1)\right) \tag{60}$$

$$\chi_n \sim (\cos(\sqrt{\lambda}y))^{(\omega/\lambda)+((\varepsilon+1)/2)}(\sin(\sqrt{\lambda}y))^{l+1+\varepsilon\nu} \times {}_2F_1\left(\frac{\omega}{\lambda}+n+l+\frac{3}{2}+\frac{\varepsilon}{2}+\varepsilon\nu, -n; l+\frac{3}{2}+\varepsilon\nu; \sin^2(\sqrt{\lambda}y)\right) \tag{61}$$

for $k = l$, and

$$W_n(y) = \left(\frac{\sqrt{\lambda}}{2}(\varepsilon+1+2n)+\frac{\omega}{\sqrt{\lambda}}\right)\tan(\sqrt{\lambda}y) - \sqrt{\lambda}(n+l+1+\nu\varepsilon)\cot(\sqrt{\lambda}y) \tag{62}$$

$$E_n = \frac{\lambda}{2}\left(2n+l+\frac{1}{2}(\varepsilon+1)\right)\left(2n+l+\frac{5}{2}+\frac{\varepsilon}{2}\right)+\omega\left(2n+l+\varepsilon k+\frac{3}{2}(\varepsilon+1)\right) +\omega\nu(1-\varepsilon)-\lambda\nu\left(2\varepsilon n+\frac{3\varepsilon}{2}+\frac{1}{2}\right) \tag{63}$$

$$\chi_n \sim (\cos(\sqrt{\lambda}y))^{(\omega/\lambda)+((\varepsilon+1)/2)}(\sin(\sqrt{\lambda}y))^{l+1-\varepsilon\nu} \times {}_2F_1\left(\frac{\omega}{\lambda}+n+l+\frac{3}{2}+\frac{\varepsilon}{2}-\varepsilon\nu, -n; l+\frac{3}{2}-\varepsilon\nu; \sin^2(\sqrt{\lambda}y)\right) \tag{64}$$

for $k = -l-1$.

4. Degeneracies

Let us take back the $D = 2$ oscillator-like interaction. If we put

$$N = 2n + l \tag{65}$$

the deformed supersymmetric energies are (cf (50))

$$E^{-}_{N_m} = \tfrac{1}{2}\lambda N(N+1) + \omega(N+m) \tag{66}$$

$$E^{+}_{N_m} = \tfrac{1}{2}\lambda (N+1)(N+2) + \omega(N+m+2) \tag{67}$$

according to the eigenvalues $\varepsilon = \pm 1$ of σ_3. It is thus clear that the degeneracies

$$E^{-}_{N_m} = E^{+}_{(N-1)_{(m-1)}} \tag{68}$$

are present in that context. Such degeneracies have to be explained [11] through operators connecting the associated eigenfunctions, i.e. (see (34), (35), (37), (38) and (51))

$$\psi^{-}_{N_m} \sim r^l (1+\lambda r^2)^{-(\omega/2\lambda)-(l/2)-(3/4)} \times {}_2F_1\left(\frac{\omega}{\lambda} + \frac{N+l+1}{2}, \frac{-N+l}{2}; l+1; \frac{\lambda r^2}{1+\lambda r^2}\right) \mathrm{e}^{\mathrm{i}m\theta} \tag{69}$$

and

$$\psi^{+}_{(N-1)_{(m-1)}} \sim r^l (1+\lambda r^2)^{-(\omega/2\lambda)-(l/2)-(3/4)} \times {}_2F_1\left(\frac{\omega}{\lambda} + \frac{N+l+1}{2}, \frac{-N+l}{2}; l; \frac{\lambda r^2}{1+\lambda r^2}\right) \mathrm{e}^{\mathrm{i}(m-1)\theta}. \tag{70}$$

These operators do exist and can be written in the form

$$A^{\pm} = \pi_1 \pm \mathrm{i}\pi_2 \pm \mathrm{i}\omega x_1 - \omega x_2 \tag{71}$$

as can be verified using a few properties of the hypergeometric functions [8].

Now, the supercharges (26) lead in this oscillator-like case to

$$Q^{\pm} = \frac{1}{\sqrt{2}}(Q_1 \pm \mathrm{i}Q_2) = (\pi_1 \mp \mathrm{i}\pi_2 \mp \mathrm{i}\omega x_1 - \omega x_2)\sigma_{\pm} \tag{72}$$

or, in other words, to

$$Q^{\pm} = A^{\mp}\sigma_{\pm}. \tag{73}$$

Consequently, the structure explaining the degeneracies (68) is nothing but the generalized SSQM-superalgebra pointed out in section 2.

Let us now turn to the $D = 3$ examples. We consider simultaneously the oscillator-like interaction ($\nu = 0$) and the Calogero interaction ($\nu \neq 0$). If we take account of (65), we can write the energies (60) as

$$E^{-}_{Nl} = \tfrac{1}{2}\lambda N(N+2) + \omega(N-l) - \lambda\nu(N-l) \tag{74}$$

$$E^{+}_{Nl} = \tfrac{1}{2}\lambda (N+1)(N+3) + \omega(N+l+3+2\nu) + \lambda\nu(N-l+1) \tag{75}$$

and the energies (63) as

$$E^{-}_{Nl} = \tfrac{1}{2}\lambda N(N+2) + \omega(N+l+1+2\nu) + \lambda\nu(N-l+1) \tag{76}$$

$$E^{+}_{Nl} = \tfrac{1}{2}\lambda (N+1)(N+3) + \omega(N-l+2) - \lambda\nu(N-l+2). \tag{77}$$

It is once again straightforward to notice that

$$E^{-}_{Nl}(k=-l-1) = E^{+}_{(N-1)(l-1)}(k=1) \tag{78}$$

and

$$E^{-}_{Nl}(k=l) = E^{+}_{(N-1)(l+1)}(k=-l-1). \tag{79}$$

These degeneracies are accidental ones by opposition to that revealed in (68). They are indeed connected with *different* values of k. The degeneracy (78) has already been observed in the undeformed case ($\lambda = 0$) by Balantekin [12] in the oscillator context and by Celka and Hussin [10] in the Calogero context. The key point is now to explain these degeneracies through *ad hoc* operators connecting

$$\begin{aligned}\psi^{-}_{Nlm}(k=-l-1) &\sim r^{l+\nu}(1+\lambda r^2)^{-(\omega/2\lambda)-(l/2)-(\nu/2)-1}\\ &\quad\times {}_2F_1\left(\frac{\omega}{\lambda}+\frac{N+l}{2}+\nu+1, \frac{-N+1}{2}; l+\nu+\frac{3}{2}; \frac{\lambda r^2}{1+\lambda r^2}\right) Y_{lm}(\theta,\varphi)\end{aligned} \tag{80}$$

and

$$\begin{aligned}\psi^{+}_{(N-1)(l-1)m'}(k=l) &\sim r^{l+\nu-1}(1+\lambda r^2)^{-(\omega/2\lambda)-(l/2)-(\nu/2)-1}\\ &\quad\times {}_2F_1\left(\frac{\omega}{\lambda}+\frac{N+l}{2}+\nu+1, \frac{-N+l}{2}; l+\nu+\frac{1}{2}; \frac{\lambda r^2}{1+\lambda r^2}\right) Y_{(l-1)m'}(\theta,\varphi)\end{aligned} \tag{81}$$

for the degeneracy (78) and

$$\begin{aligned}\psi^{-}_{Nlm}(k=l) &\sim r^{l-\nu}(1+\lambda r^2)^{-(\omega/2\lambda)-(l/2)+(\nu/2)-1}\\ &\quad\times {}_2F_1\left(\frac{\omega}{\lambda}+\frac{N+l}{2}-1-\nu, \frac{-N+l}{2}; l-\nu+\frac{3}{2}; \frac{\lambda r^2}{1+\lambda r^2}\right) Y_{lm}(\theta,\varphi)\end{aligned} \tag{82}$$

and

$$\begin{aligned}\psi^{+}_{(N-1)(l+1)m'}(k=-l-1) &\sim r^{l-\nu+1}(1+\lambda r^2)^{-(\omega/2\lambda)-(l/2)-(\nu/2)-2}\\ &\quad\times {}_2F_1\left(\frac{\omega}{\lambda}+\frac{N+l}{2}-\nu+2, \frac{-N+l}{2}; l-\nu+\frac{5}{2}; \frac{\lambda r^2}{1+\lambda r^2}\right) Y_{(l+1)m'}(\theta,\varphi)\end{aligned} \tag{83}$$

for the degeneracy (79). Because we are in the central context, the spherical harmonics $Y_{ln}(\theta,\varphi)$ do not play any significant role and it is thus sufficient to try to relate the radial parts of the functions (80)–(83). Using the properties of the hypergeometric function [8], it is possible to verify that the operators

$$Q^- = \left(-(1+\lambda r^2)\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l+1+\nu}{r} + \omega r\right)\sigma_- \qquad Q^+ = (Q^-)^\dagger \tag{84}$$

do connect the functions (80) and (81), while the operators

$$P^- = \left(-(1+\lambda r^2)\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\nu-l}{r} + \omega r\right)\sigma_- \qquad P^+ = (P^-)^\dagger \tag{85}$$

relate the functions (82) and (83). It has to be noticed that these operators do not generate a closed Lie structure.

Acknowledgment

It is a pleasure to thank Professor J Beckers for a careful reading of the manuscript.

Appendix

We recall here, in order to be self-consistent, how to determine the eigenvalues and eigenfunctions of the $(D = 1)$-dimensional Hamiltonian

$$H_0 = \tfrac{1}{2} p^2 + V_0(x) \tag{86}$$

by using the factorization method [13]. The first step is to rewrite the equation

$$H_0 \psi = E\psi \tag{87}$$

in the form

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2} W_0^2(x) - \frac{1}{2}\frac{\mathrm{d}W_0}{\mathrm{d}x} + E_0\right)\psi = 0 \tag{88}$$

where E_0 is the lowest energy corresponding to V_0. In other words, we have to solve the Riccatti equation

$$V_0(x) = \frac{1}{2} W_0^2(x) - \frac{1}{2}\frac{\mathrm{d}W_0}{\mathrm{d}x} + E_0 \tag{89}$$

and then obtain W_0, E_0 and, consequently, the fundamental eigenfunction (up to a normalization factor)

$$\psi_0 \sim \exp\left(-\int W_0 \,\mathrm{d}x\right). \tag{90}$$

The other steps are then based on the recurrence relation [13]

$$W_{n-1}^2 + \frac{\mathrm{d}W_{n-1}}{\mathrm{d}x} + 2E_{n-1} = W_n^2 - \frac{\mathrm{d}W_n}{\mathrm{d}x} + 2E_n \qquad n = 1, 2, \ldots \tag{91}$$

leading to the knowledge of W_n and E_n. The corresponding eigenfunctions are obtained through

$$\psi_n \sim \left(\frac{\mathrm{d}}{\mathrm{d}x} - W_0\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_1\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_{n-1}\right)\exp\left(-\int W_n \,\mathrm{d}x\right). \tag{92}$$

We can thus get the whole set of energy eigenvalues and eigenfunctions of (86).

References

[1] For a review, see Lahiri A, Roy P K and Bagchi B 1990 *Int. J. Mod. Phys.* A **5** 1383
Cooper F, Khare A and Sukhatme 1995 *Phys. Rep.* **251** 267
[2] Witten E 1981 *Nucl. Phys.* B **188** 513
[3] Beckers J, Debergh N, Hussin V and Sciarrino A 1990 *J. Phys. A: Math. Gen.* **23** 3647
[4] Beckers J, Dehin D and Hussin V 1987 *J. Phys. A: Math. Gen.* **20** 1137
[5] Abdesselam B, Beckers J, Chakrabarti A and Debergh N 1996 *J. Phys. A: Math. Gen.* **29** 3075
[6] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309
Leemon H I 1979 *J. Phys. A: Math. Gen.* **12** 489
[7] Kostelecký V A and Russel N 1996 *J. Math. Phys.* **37** 2166
[8] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* 3rd edn (Berlin: Springer)
[9] Daskaloyannis C 1992 *J. Phys. A: Math. Gen.* **25** 2261
[10] Celka P and Hussin V 1987 *Mod. Phys. Lett.* A **2** 391
[11] Moshinski M, Quesne C and Loyola G 1990 *Ann. Phys.* **198** 103
[12] Balantekin A B 1985 *Ann. Phys.* **164** 277
[13] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
Beckers J and Debergh N 1997 *J. Nonlinear Math. Phys.* **4** 401